The Analogous Acoustical Impedance for Discontinuities and Constrictions of Circular Cross Section*

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The impedance introduced by an abrupt change of circular cross section of a tube has been examined. It is shown that in the case of an ideal viscousless fluid the effect of the discontinuity is to introduce an inductance in series with the acoustical transmission system. The discontinuity inductance has been determined as a function of the ratio of the tube radii and has been plotted for different values of this ratio. The problem of a small tube or constriction inserted between two larger tubes has also been treated. It is shown that the constriction inductance is equal to the sum of the discontinuity inductances of each end taken separately to a very good approximation. The constriction inductance can be considered as a correction term to be added to the analogous acoustical inductance of the tube and can be interpreted physically as an increase in the equivalent length of the tube.

I. INTRODUCTION

WHEN two tubes of different circular cross section are joined together to form an acoustical transmission system, an additional impedance is introduced owing to the abrupt change of circular cross section at the tube junction. This impedance, which is shown to be an inductance, will be called the discontinuity inductance and has been determined as a function of the ratio of the tube radii. The analysis is based upon the assumption that the wavelength is long in comparison with the tube radii. If the ratio of tube radii is unity, there is no change in circular cross section and the discontinuity inductance is zero. If the ratio of tube radii is zero, corresponding to an open tube fitted with an infinite flange, the discontinuity inductance is (ρ/A) $(8r/3\pi)$.

The work has been extended to include the case of a small tube or constriction inserted between two larger tubes. It is shown that the constriction inductance, which is defined to be the inductance introduced by the change in tube cross section at both ends of the constriction, is equal to the sum of the discontinuity inductances of each end taken separately to a very good approximation. It should be noted that the constriction inductance can be considered as a correction term to be added to the analogous acoustical inductance of the tube and can be interpreted physically as an increase in the equivalent length of the tube.

The theoretical approach to the problem consists of setting up solutions satisfying both the wave equation and boundary conditions for each region separately. The solutions for each region are then required to satisfy the same boundary conditions across the common boundary. It is possible to interpret the results in such a way that the discontinuity can be represented by a lumped series inductance located at the junction.

The matching technique used in this paper was first applied to electromagnetic cavity resonator problems by Hahn.[1] Whinnery and Jamieson[2] have used the same method to study the effect of discontinuities in electrical transmission systems and have shown that the discontinuity can be represented by a simple equivalent circuit with lumped circuit elements.

Miles[3] has considered the reflection of sound due to a change in cross section of a circular tube and has determined expressions for the reflection and transmission coefficients. No numerical calculations are given. In another paper[4] Miles studied the effect of a plane discontinuity on a plane wave propagated in a cylindrical tube of arbitrary cross section by using a variational technique. He also developed a transmission line analogy. In Miles' treatment of the transmission line analogy, however, his choice of voltage and current is inverted from the elementary electrical analogy found in the standard literature.[5] He sets the particle velocity proportional to the voltage and the pressure proportional to the current in order to simplify certain boundary conditions. With this transmission line analogy the effect of a plane discontinuity on a plane wave is represented by a capacitance at the discontinuity. No numerical results are given.

II. THE EQUIVALENT CIRCUIT

Consider an acoustical transmission system of the kind shown in Fig. 1. It is desired to represent it by an equivalent transmission line and a lumped impedance at the discontinuity. It should be noted that an equivalent circuit can be drawn only for a particular mode of the transmission system and not for the transmission system itself. Therefore, we restrict ourselves to the case in which the only propagating mode is the zeroth order mode. This restriction means that the transverse dimensions of the acoustical transmission system must

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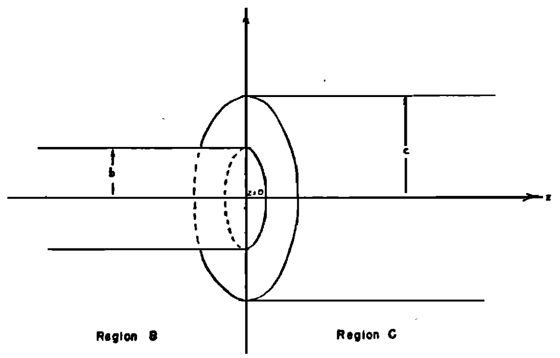

FIG. 1. Cylindrical plane discontinuity.

be small compared to the wavelength, so that all higher order modes set up at the discontinuity will be attenuated rapidly with distance.

The physical behavior in regions B and C and at the discontinuity will now be discussed. In region B, for $z<-\epsilon$, only the zeroth order mode exists. There is a wave traveling to the right and another traveling to the left (the reflected wave). In region C, for $z>+\epsilon$, only the zeroth order mode exists. This wave is propagated to the right. It is assumed there is no termination; consequently, there is no reflected wave. At the discontinuity, $z=0$, higher order modes exist. These, however, are attenuated extremely rapidly unless the transverse dimensions of the acoustic transmission system are appreciable compared to the wavelength. Therefore, in the neighborhood of the discontinuity, higher order modes as well as the zeroth order mode must be considered. On mathematical grounds it is necessary to consider the higher order modes in the neighborhood of the discontinuity in order to satisfy the boundary conditions.

Near the discontinuity

$$P=P_0+P_H,$$
$$V=V_0+V_H,$$

where P and V are the total pressure and total volume flow, P_0 and V_0 are the pressure and volume flow due to the zeroth order mode, and P_H and V_H are the pressure and volume flow due to the higher order modes. If the volume flow is defined as the integral of the normal component of the velocity across the tube cross section

$$V=\int_S \mathbf{v}\cdot\mathbf{n}dS, \tag{1}$$

it can be shown from the distribution of all higher order modes that their contribution to the volume flow integral is zero. Therefore,

$$V_H=0$$

or

$$V=V_0.$$

We conclude that total volume flow is exactly the zeroth order mode. Since total volume flow must be continuous across the discontinuity, the volume flow in the zeroth order mode must also be continuous. Therefore, at the discontinuity separating regions B and C

$$(V_{0B})_{z=0}=(V_{0C})_{z=0}. \tag{2}$$

The contribution to the pressure from the higher order modes P_H is not zero. Continuity of total pressure at the discontinuity can be written as

$$(P_{0C})_{z=0}-(P_{0B})_{z=0}=(P_{HB})_{z=0}-(P_{HC})_{z=0}. \tag{3}$$

Evidently the transmission line pressure is discontinuous by an amount given by the difference in the pressures because of the higher order modes in the two regions. The continuity of volume flow and discontinuity of pressure at $z=0$ may be compensated for in the transmission line equivalent circuit by inserting an impedance in series with the line at the discontinuity. This lumped impedance has the value

$$Z=\frac{(P_{HC})_{z=0}-(P_{HB})_{z=0}}{(V_0)_{z=0}}=\frac{(P_{0B})_{z=0}-(P_{0C})_{z=0}}{(V_0)_{z=0}}. \tag{4}$$

The equivalent circuit is shown in Fig. 2.

III. THEORY

Consider an acoustical transmission system in which a tube of radius b is joined to a tube of radius c, where c is greater than b and the change in cross section is abrupt. Let us introduce a cylindrical coordinate system oriented in such a way that the z axis coincides with the axes of the tubes and the origin is taken in the plane of the discontinuity. (See Fig. 1.) Suppose that the tube in region B is excited by the zeroth order mode and that the tube in region C is terminated in its characteristic impedance so that there is no reflection. Let $B_0{}^1$ be the amplitude of the incident zeroth order mode in region B, $B_0{}^2$ be the amplitude if the reflected zeroth order mode in region B, and C_0 be the amplitude of the transmitted zeroth order mode in region C. At the discontinuity, higher order modes are excited but are attenuated rapidly if the transverse dimensions of the tubes are small compared to the wavelength. Let B_n and C_n be the amplitudes of the attenuated modes in regions B and C, respectively.

The expressions for the pressure in regions B and C are given by

$$p_B=B_0{}^1e^{+ikz}+B_0{}^2e^{-ikz}+\sum_{n=1}^{\infty}B_nJ_0(\pi\alpha_{0n}r/b)\exp(+\gamma_{0n}'z) \tag{5}$$

and

$$p_C=C_0e^{+ikz}+\sum_{n=1}^{\infty}C_mJ_0(\pi\alpha_{0m}r/c)\exp(-\gamma_{0m}z). \tag{6}$$

In Eqs. (5) and (6)

$$\left.\begin{aligned} c' &= \text{velocity of sound} \\ k &= \omega/c' = 2\pi/\lambda \\ \gamma'_{0n} &= (\pi\alpha_{0n}/b)[1-(2b/\lambda\alpha_{0n})^2]^{\frac{1}{2}} \\ \gamma_{0m} &= (\pi\alpha_{0m}/c)[1-(2c/\lambda\alpha_{0m})^2]^{\frac{1}{2}} \end{aligned}\right\}, \tag{7}$$

and the time factor $\exp(-i\omega t)$ has been omitted for convenience. Note that z is negative in region B and positive in region C. $\pi\alpha_{0q}$ are roots of the equation

$$J_1(\pi\alpha_{0q})=0 \tag{8}$$

and result from the fact that the walls are assumed to be rigid, that is, the radial component of velocity is zero. It can easily be verified that the preceding mode solutions satisfy the scalar wave equation for the sound pressure.

We shall find it convenient to introduce the following specific acoustic admittances:

$$Y_{B_n}=+\frac{1}{i\omega\rho p_{B_n}}\frac{\partial p_{B_n}}{\partial z}=+\frac{\gamma_{0n}'}{i\omega\rho},\quad n\neq 0 \tag{9}$$

$$Y_{C_m}=+\frac{1}{i\omega\rho p_{C_m}}\frac{\partial p_{C_m}}{\partial z}=-\frac{\gamma_{0m}}{i\omega\rho},\quad m\neq 0. \tag{10}$$

Let us next write expressions for the normal components of velocity in both regions. We have, making use of (9) and (10),

$$v_{zB}=\frac{1}{i\omega\rho}\frac{\partial p_B}{\partial z}=+\frac{k}{\omega\rho}B_0^1e^{+ikz}-\frac{k}{\omega\rho}B_0^2e^{-ikz} + \sum_{n=1}^{\infty}Y_{B_n}B_nJ_0(\pi\alpha_{0n}r/b)\exp(+\gamma_{0n}'z) \tag{11}$$

and

$$v_{zC}=\frac{1}{i\omega\rho}\frac{\partial p_C}{\partial z}=+\frac{k}{\omega\rho}C_0e^{+ikz} + \sum_{m=1}^{\infty}Y_{C_m}C_mJ_0(\pi\alpha_{0m}r/c)\exp(-\gamma_{0m}z). \tag{12}$$

The boundary conditions at the discontinuity are the following:

$$(p_B)_{z=0}=(p_C)_{z=0},\quad 0<r<b \tag{13}$$

$$\frac{1}{i\omega\rho}\left(\frac{\partial p_B}{\partial z}\right)_{z=0}=\frac{1}{i\omega\rho}\left(\frac{\partial p_C}{\partial z}\right)_{z=0},\quad 0<r<b \tag{14}$$

$$\frac{1}{i\omega\rho}\left(\frac{\partial p_C}{\partial z}\right)_{z=0}=0,\quad b<r<c. \tag{15}$$

In the above p_B and p_C are the total pressures, that is, the sum of transmission line and higher order modes.

The expressions for the total pressure in the two regions when $z=0$ are given by

$$(p_B)_{z=0}=B_0^1+B_0^2+\sum_{n=1}^{\infty}B_nJ_0(\pi\alpha_{0n}r/b)$$

or

$$(p_B)_{z=0}=B_0+\sum_{n=1}^{\infty}B_nJ_0(\pi\alpha_{0n}r/b) \tag{16}$$

and

$$(p_C)_{z=0}=C_0+\sum_{m=1}^{\infty}C_mJ_0(\pi\alpha_{0m}r/c), \tag{17}$$

where the amplitudes of the transmission line modes have been added to give a single term.

The expressions for the normal velocity components in the two regions when $z=0$ are given by

$$\frac{1}{i\omega\rho}\left(\frac{\partial p_B}{\partial z}\right)_{z=0}=+\frac{k}{\omega\rho}B_0^1-\frac{k}{\omega\rho}B_0^2 + \sum_{n=1}^{\infty}Y_{B_n}B_nJ_0(\pi\alpha_{0n}r/b)$$

or

$$\frac{1}{i\omega\rho}\left(\frac{\partial p_B}{\partial z}\right)_{z=0}=Y_{B_0}B_0+\sum_{n=1}^{\infty}Y_{B_n}B_nJ_0(\pi\alpha_{0n}r/b) \tag{18}$$

and

$$\frac{1}{i\omega\rho}\left(\frac{\partial p_C}{\partial z}\right)_{z=0}=+\frac{k}{\omega\rho}C_0+\sum_{m=1}^{\infty}Y_{C_m}C_mJ_0(\pi\alpha_{0m}r/c)$$

or

$$\frac{1}{i\omega\rho}\left(\frac{\partial p_C}{\partial z}\right)_{z=0}=Y_{C_0}C_0+\sum_{m=1}^{\infty}Y_{C_m}C_mJ_0(\pi\alpha_{0m}r/c). \tag{19}$$

Consider boundary conditions (14) and (15). Let us first find an expression for the $Y_{B_0}B_0$ term. We have from (14)

$$\int_0^b\left(\frac{\partial p_B}{\partial z}\right)_{z=0}rdr=\int_0^b\left(\frac{\partial p_C}{\partial z}\right)_{z=0}rdr. \tag{20}$$

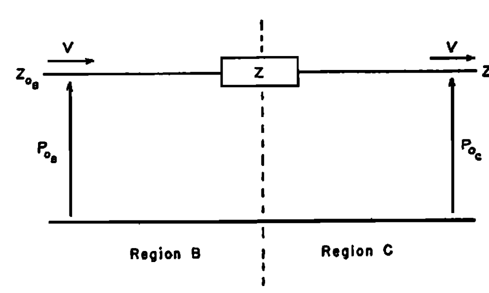

FIG. 2. The equivalent circuit.

From (15) we obtain the integral

$$\int_b^c \left(\frac{\partial p_C}{\partial z}\right)_{z=0} r dr$$

which has the value zero. This is added to the left-hand side of (20) yielding the following expression:

$$\int_0^b \left(\frac{\partial p_B}{\partial z}\right)_{z=0} r dr = \int_0^c \left(\frac{\partial p_C}{\partial z}\right)_{z=0} r dr. \qquad (21)$$

Substituting from (18) and (19) we have

$$\int_0^b Y_{B_0} B_0 r dr + \sum_{n=1}^{\infty} Y_{B_n} B_n \int_0^b J_0(\pi\alpha_{0n} r/b) r dr$$
$$= \int_0^c Y_{C_0} C_0 r dr + \sum_{m=1}^{\infty} Y_{C_m} C_m \int_0^c J_0(\pi\alpha_{0m} r/c) r dr$$

or

$$b^2 Y_{B_0} B_0 = c^2 Y_{C_0} C_0. \qquad (22)$$

We perform a similar integration in order to obtain the higher order coefficients. We multiply both sides by $rJ_0(\pi\alpha_{0p}r/c)$ instead of r, however, before integrating.

$$\int_0^b \left(\frac{\partial p_B}{\partial z}\right)_{z=0} J_0(\pi\alpha_{0p} r/c) r dr$$
$$= \int_0^c \left(\frac{\partial p_C}{\partial z}\right)_{z=0} J_0(\pi\alpha_{0p} r/c) r dr. \qquad (23)$$

Substituting from (18) and (19) we have

$$\int_0^b Y_{B_0} B_0 J_0(\pi\alpha_{0p} r/c) r dr$$
$$+ \sum_{n=1}^{\infty} Y_{B_n} B_n \int_0^b J_0(\pi\alpha_{0n} r/b) J_0(\pi\alpha_{0p} r/c) r dr$$
$$= \int_0^c Y_{C_0} C_0 J_0(\pi\alpha_{0p} r/c) r dr$$
$$+ \sum_{m=1}^{\infty} Y_{C_m} C_m \int_0^c J_0(\pi\alpha_{0m} r/c) J_0(\pi\alpha_{0p} r/c) r dr. \qquad (24)$$

Letting $p=m$ and using several well-known integrals we obtain

$$Y_{B_0} B_0 \left(\frac{bc}{\pi\alpha_{0m}}\right) J_1(\pi\alpha_{0m} b/c)$$
$$+ \sum_{n=1}^{\infty} \frac{(\pi\alpha_{0m} b/c) Y_{B_n} B_n}{(\pi\alpha_{0m}/c)^2 - (\pi\alpha_{0n}/b)^2} J_0(\pi\alpha_{0n}) J_1(\pi\alpha_{0m} b/c)$$
$$= Y_{C_m} C_m \left(\frac{c^2}{2}\right) J_0^2(\pi\alpha_{0m}). \qquad (25)$$

Rearranging the above we have

$$\frac{Y_{C_m} C_m}{Y_{B_0} B_0} = 2\alpha \frac{J_1(x_m\alpha)}{x_m J_0^2(x_m)}$$
$$+ 2\alpha^2 \sum_{n=1}^{\infty} \frac{Y_{B_n} B_n}{Y_{B_0} B_0} \frac{T_{mn}(\alpha) J_0^2(x_n)}{x_n J_0^2(x_m)}, \qquad (26)$$

where

$$\alpha = b/c, \qquad (27)$$

$$x_m = \pi\alpha_{0m}, \qquad (28)$$

$$T_{mn}(\alpha) = \frac{(x_m\alpha/x_n)}{[(x_m\alpha/x_n)^2 - 1]} \frac{J_1(x_m\alpha)}{J_0(x_n)}. \qquad (29)$$

We shall now consider boundary condition (13). Let us first find an expression for the B_0 term. We have

$$\int_0^b (p_B)_{z=0} r dr = \int_0^b (p_C)_{z=0} r dr. \qquad (30)$$

Substituting from (16) and (17) we obtain

$$\int_0^b B_0 r dr + \sum_{n=1}^{\infty} B_n \int_0^b J_0(\pi\alpha_{0n} r/b) r dr$$
$$= \int_0^b C_0 r dr + \sum_{m=1}^{\infty} C_m \int_0^b J_0(\pi\alpha_{0m} r/c) r dr$$

or

$$\frac{b^2}{2} B_0 = \frac{b^2}{2} C_0 + \sum_{m=1}^{\infty} \left(\frac{bc}{\pi\alpha_{0m}}\right) C_m J_1(\pi\alpha_{0m} b/c). \qquad (31)$$

When we rearrange and use (27) and (28), the above becomes

$$B_0 = C_0 + 2\sum_{m=1}^{\infty} C_m \frac{J_1(x_m\alpha)}{x_m\alpha}. \qquad (32)$$

We shall next obtain the B_n coefficient. Multiplying both sides of (13) by $J_0(\pi\alpha_{0p}/b)r$ and integrating from 0 to b with respect to r, we obtain

$$\int_0^b (p_B)_{z=0} J_0(\pi\alpha_{0p} r/b) r dr$$
$$= \int_0^b (p_C)_{z=0} J_0(\pi\alpha_{0p} r/b) r dr. \qquad (33)$$

By substituting from (16) and (17) we have

$$\int_0^b B_0 J_0(\pi\alpha_{0p} r/b) r dr$$
$$+ \sum_{n=1}^{\infty} B_n \int_0^b J_0(\pi\alpha_{0n} r/b) J_0(\pi\alpha_{0p} r/b) r dr$$
$$= \int_0^b C_0 J_0(\pi\alpha_{0p} r/b) r dr$$
$$+ \sum_{m=1}^{\infty} C_m \int_0^b J_0(\pi\alpha_{0m} r/c) J_0(\pi\alpha_{0p} r/b) r dr. \qquad (34)$$

Letting $p=n$ and using several well-known integrals, we obtain

$$B_n\left(\frac{b^2}{2}\right)J_0^2(\pi\alpha_{0n})$$
$$=-\sum_{m=1}^{\infty}C_m\frac{b(\pi\alpha_{0m}/c)}{(\pi\alpha_{0n}/b)^2-(\pi\alpha_{0m}/c)^2}J_0(\pi\alpha_{0n})J_1(\pi\alpha_{0m}b/c).$$

By rearranging, and using (27), (28), and (29), the above equation becomes

$$B_n=2\sum_{m=1}^{\infty}C_m\frac{T_{mn}(\alpha)}{x_n}. \tag{35}$$

From the results already obtained, it is possible to determine an infinite number of inhomogeneous simultaneous equations in terms of the infinite number of unknown ratios $(Y_{B_p}B_p/Y_{B_0}B_0)$. These ratios will be needed in order to determine the exact value of the discontinuity inductance. Now

$$Y_{B_n}=+\frac{\gamma'_{0n}}{i\omega\rho}\approx\frac{x_n}{i\omega\rho b}, \tag{36}$$

$$Y_{C_m}=-\frac{\gamma_{0m}}{i\pi\rho}\approx-\frac{x_m}{i\omega\rho c}, \tag{37}$$

provided the wavelength is long in comparison to the tube radii. Substituting into (35) we have

$$B_n\left(Y_{B_n}\frac{b}{x_n}\right)=2\sum_{m=1}^{\infty}C_m\frac{T_{mn}(\alpha)}{x_n}\left(-Y_{C_m}\frac{c}{x_m}\right)$$

or

$$\frac{Y_{B_p}B_p}{Y_{B_0}B_0}=-\frac{2}{\alpha}\sum_{m=1}^{\infty}\frac{Y_{C_m}C_m}{Y_{B_0}B_0}\frac{T_{mp}(\alpha)}{x_m}. \tag{38}$$

Equation (26) can now be substituted directly into (38). We obtain

$$\frac{Y_{B_p}B_p}{Y_{B_0}B_0}=-4\sum_{m=1}^{\infty}\frac{T_{mp}(\alpha)}{x^2_m}\frac{J_1(x_m\alpha)}{J_0^2(x_m)}$$
$$-4\alpha\sum_{m=1}^{\infty}\frac{T_{mp}(\alpha)}{x_mJ_0^2(x_m)}\sum_{n=1}^{\infty}\frac{Y_{B_n}B_n}{Y_{B_0}B_0}\frac{T_{mn}(\alpha)}{x_n}J_0^2(x_n). \tag{39}$$

Equation (39) represents an infinite number of inhomogeneous simultaneous equations, one for each value of p, in the infinite number of unknown ratios. It is possible to evaluate a finite number of the unknown ratios by taking the same number of simultaneous equations.

Before proceeding with a discussion of the discontinuity inductance, let us first verify Eq. (2) given in the introduction. Using the definition of volume flow given by (1), the volume flow for region C is

$$(V_C)_{z=0}=\frac{1}{i\omega\rho}\int_0^c\left(\frac{\partial p_C}{\partial z}\right)_{z=0}2\pi r dr. \tag{40}$$

Substituting from Eq. (19) we have

$$(V_C)_{z=0}=2\pi Y_{C_0}C_0\int_0^c r dr$$
$$+2\pi\sum_{m=1}^{\infty}Y_{C_m}C_m\int_0^c J_0(\pi\alpha_{0m}r/c)r dr$$

or

$$(V_C)_{z=0}=\pi c^2Y_{C_0}C_0. \tag{41}$$

We obtain a similar expression for the volume flow in region B. Thus,

$$(V_B)_{z=0}=\pi b^2Y_{B_0}B_0. \tag{42}$$

By using Eq. (22) we see that

$$(V_C)_{z=0}=\pi c^2Y_{C_0}C_0=\pi b^2Y_{B_0}B_0$$

or

$$(V_B)_{z=0}=(V_C)_{z=0}. \tag{43}$$

It should be noted that even if we do not restrict ourselves to the plane of the discontinuity, the contribution to volume flow from the higher order modes is still zero. This proves the statement made in the introduction that the total volume flow at any point is that of the zeroth order mode.

IV. THE DISCONTINUITY INDUCTANCE

The zeroth order mode pressures in the two regions at the discontinuity are

$$(p_{0B})_{z=0}=B_0 \tag{44}$$

and

$$(p_{0C})_{z=0}=C_0. \tag{45}$$

Equation (32) indicates, however, that the pressure in the zeroth order modes in the two regions is not continuous across the change of cross section. Thus,

$$(p_{0B})_{z=0}=(p_{0C})_{z=0}+2\sum_{m=1}^{\infty}C_m\frac{J_1(x_m\alpha)}{x_m\alpha}. \tag{46}$$

This discontinuity of pressure may be represented by a lumped impedance at the change of cross section. Since

$$p=VZ_\alpha,$$

where Z_α is an analogous acoustical impedance, Eq. (46) can be written as

$$(p_{0B})_{z=0}-(p_{0C})_{z=0}=VZ_\alpha$$

or

$$(p_{0B})_{z=0}-(p_{0C})_{z=0}=\pi b^2Y_{B_0}B_0Z_\alpha,$$

where we have used (42). Comparing the above result with (46) we obtain

$$Z_\alpha=\frac{2}{\pi b^2}\sum_{m=1}^{\infty}\frac{C_m}{Y_{B_0}B_0}\frac{J_1(x_m\alpha)}{x_m\alpha}. \tag{47}$$

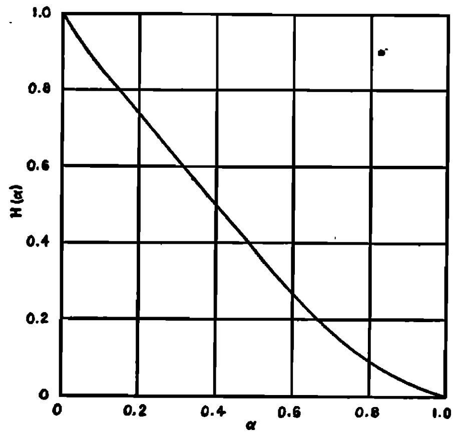

FIG. 3. Discontinuity inductance correction factor $H(\alpha)$ *versus* ratio of tube radii α.

Substituting from (7) and (10) into (47) we have

$$Z_\alpha=-i\omega\frac{2\rho}{\pi b}\sum_{m=1}^{\infty}\left(\frac{Y_{C_m}C_m}{Y_{C_0}C_0}\right)\frac{J_1(x_m\alpha)}{x^2{}_m\alpha^2} \tag{48}$$

or

$$L(\alpha)=\frac{2\rho}{\pi b}\sum_{m=1}^{\infty}\left(\frac{Y_{C_m}C_m}{Y_{C_0}C_0}\right)\frac{J_1(x_m\alpha)}{x^2{}_m\alpha^2}, \tag{49}$$

where $L(\alpha)$ is the analogous acoustical inductance. If Eq. (26) is substituted into (49), we obtain the desired result for the discontinuity inductance:

$$L(\alpha)=\frac{8\rho}{3\pi^2 b}\bigg\{H(\alpha)+\frac{3\pi}{2}\sum_{m=1}^{\infty}\frac{J_1(x_m\alpha)}{x^2{}_mJ^2{}_0(x_m)}\sum_{n=1}^{\infty}\left(\frac{Y_{B_n}B_n}{Y_{B_0}B_0}\right)\frac{T_{mn}(\alpha)J_0{}^2(x_n)}{x_n}\bigg\}, \tag{50}$$

$$H(\alpha)=\frac{3\pi}{2}\sum_{m=1}^{\infty}\frac{J_1{}^2(x_m\alpha)}{(x_m\alpha)[x_mJ_0(x_m)]^2}. \tag{51}$$

The unknown constants appearing in (50) can be found from (39). For most practical applications, however, the analogous inductance given by the approximate expression

$$L(\alpha)=(8\rho/3\pi^2b)H(\alpha) \tag{52}$$

is sufficiently accurate. Values of $H(\alpha)$ have been plotted as a function of α, the ratio of the tube radii, in Fig. 3. $H(\alpha)$ will be called the discontinuity inductance correction factor.

The analysis presented in the preceding pages is based upon the assumption that the wavelength is long in comparison with the tube radii. Note that if the ratio of tube radii is unity, the discontinuity inductance is zero. This result is physically obvious since there is no change in tube cross section and consequently no additional inductance is introduced. If the ratio of tube radii is zero, corresponding to an open tube fitted with an infinite flange, the discontinuity inductance is $(8\rho/3\pi^2b)$. This is the same result obtained by Morse.[6] His analysis, however, is based on the assumption that the layer of air at the open end of the tube vibrates in the same way as a circular piston set flush in an infinite plane wall.

The discontinuity considered in the preceding discussion is one in which there is an increase in tube cross section at the junction. For discontinuities in which there is a decrease in tube cross section at the junction, it can be shown that the expression for the discontinuity inductance is the same as Eq. (52), provided b is taken to be the radius of the smaller tube and α is the ratio of the small tube to the larger tube.

V. THE CONSTRICTION INDUCTANCE

Consider an acoustical transmission system in which a small tube or constriction of radius b is inserted between two larger tubes of radii a and c. Let us introduce a cylindrical coordinate system oriented in such a way that the z axis coincides with the axes of the tubes and the origin is taken in the plane of the discontinuity occurring at the junction between regions A and B. See Fig. 4. Suppose that the tube in region A is excited by the zeroth order mode and that the tube in region C is terminated in its characteristic impedance so that there is no reflection. Let $A_0{}^1$ and $A_0{}^2$ be the amplitudes of the incident and reflected zeroth order modes in region A, $B_0{}^1$ and $B_0{}^2$ be the amplitudes of the incident and reflected zeroth order modes in region B, and C_0 be the amplitude of the transmitted zeroth order in region C. Also, let A_p, B_n, and C_m be the amplitudes of the higher order modes in regions A, B, and C, respectively. The solution of the constriction problem follows the same general procedure used previously. For this reason only the essential steps will be given.

The expressions for the pressure and normal component of velocity at the junction between regions A and B are

$$(p_A)_{z=0}=A_0+\sum_{p=1}^{\infty}A_pJ_0(\pi\alpha_{0p}r/a), \tag{53}$$

$$\frac{1}{i\omega\rho}\left(\frac{\partial p_A}{\partial z}\right)_{z=0}=Y_{A_0}A_0+\sum_{p=1}^{\infty}Y_{A_p}A_pJ_0(\pi\alpha_{0p}r/a), \tag{54}$$

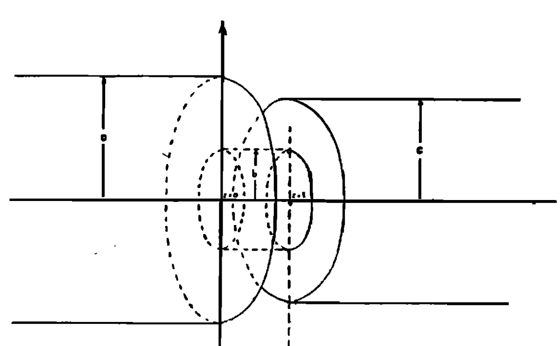

FIG. 4. Constriction discontinuity.

[6] See reference 5, p. 247.

and

$$(p_B)_{z=0}=B_0+\sum_{n=1}^{\infty}B_nJ_0(\pi\alpha_{0n}r/b), \quad (55)$$

$$\frac{1}{i\omega\rho}\left(\frac{\partial p_B}{\partial z}\right)_{z=0}=\frac{B_0^*}{c'\rho}+\sum_{n=1}^{\infty}Y_{B_n}B_n^*J_0(\pi\alpha_{0n}r/b). \quad (56)$$

In the above expressions

$$\left.\begin{aligned} k&=2\pi/\lambda\\ \gamma_{0p}&=(\pi\alpha_{0p}/a)[1-2a/\lambda\alpha_{0p})^2]^{\frac{1}{2}}\\ \gamma_{0n}'&=(\pi\alpha_{0n}/b)[1-(2b/\lambda\alpha_{0n})^2]^{\frac{1}{2}}\\ \gamma_{0m}''&=(\pi\alpha_{0m}/c)[1-(2c/\lambda\alpha_{0m})^2]^{\frac{1}{2}}\end{aligned}\right\}, \quad (57)$$

and

$$\left.\begin{aligned} Y_{A_p}&=+\frac{\gamma_{0p}}{i\omega\rho}\approx+\frac{\pi\alpha_{0p}}{i\omega\rho a}, \quad p\neq 0\\ Y_{B_n}&=-\frac{\gamma_{0n}'}{i\omega\rho}\approx-\frac{\pi\alpha_{0n}}{i\omega\rho b}, \quad n\neq 0\\ Y_{C_m}&=-\frac{\gamma_{0m}''}{i\omega\rho}\approx-\frac{\pi\alpha_{0m}}{i\omega\rho c}, \quad m\neq 0\end{aligned}\right\}. \quad (58)$$

We have also made the following substitutions:

$$\left.\begin{aligned} A_0&=A_0^1+A_0^2\\ Y_{A_0}A_0&=\frac{1}{\rho c'}(A_0^1-A_0^2)\\ B_0&=B_0^1+B_0^2\\ B_n&=B_n^1+B_n^2\\ B_0^*&=B_0^1-B_0^2\\ B_n^*&=B_n^1-B_n^2\end{aligned}\right\}. \quad (59)$$

The expressions for the pressure and normal component of velocity at the junction between regions B and C are

$$(p_B)_{z=l}=\mathfrak{B}_0+\sum_{n=1}^{\infty}\mathfrak{B}_nJ_0(\pi\alpha_{0n}r/b), \quad (60)$$

$$\frac{1}{i\omega\rho}\left(\frac{\partial p_B}{\partial z}\right)_{z=l}=\frac{\mathfrak{B}_0^*}{c'\rho}+\sum_{n=1}^{\infty}Y_{B_n}\mathfrak{B}_n^*J_0(\pi\alpha_{0n}r/b), \quad (61)$$

and

$$(p_C)_{z=l}=\mathfrak{C}_0+\sum_{m=1}^{\infty}\mathfrak{C}_mJ_0(\pi\alpha_{0m}r/c), \quad (62)$$

$$\frac{1}{i\omega\rho}\left(\frac{\partial p_C}{\partial z}\right)_{z=l}=Y_{C_0}\mathfrak{C}_0+\sum_{m=1}^{\infty}Y_{C_m}\mathfrak{C}_mJ_0(\pi\alpha_{0m}r/c), \quad (63)$$

where

$$\left.\begin{aligned} \mathfrak{B}_0&=B_0^1e^{+ikl}+B_0^2e^{-ikl}\\ \mathfrak{B}_n&=B_n^1e^{-\gamma'_{0n}l}+B_n^2e^{+\gamma'_{0n}l}\\ \mathfrak{B}_0^*&=B_0^1e^{+ikl}-B_0^2e^{-ikl}\\ \mathfrak{B}_n^*&=B_n^1e^{-\gamma'_{0n}l}-B_n^2e^{+\gamma'_{0n}l}\\ \mathfrak{C}_0&=C_0e^{+ikl}\\ \mathfrak{C}_m&=C_me^{-\gamma''_{0m}l}\\ Y_{C_0}\mathfrak{C}_0&=\frac{1}{\rho c'}C_0e^{+ikl}\end{aligned}\right\}. \quad (64)$$

The boundary conditions at $z=0$ are

$$(p_A)_{z=0}=(p_B)_{z=0}, \quad 0<r<b \quad (65)$$

$$\frac{1}{i\omega\rho}\left(\frac{\partial p_A}{\partial z}\right)_{z=0}=\frac{1}{i\omega\rho}\left(\frac{\partial p_B}{\partial z}\right)_{z=0}, \quad 0<r<b \quad (66)$$

$$\frac{1}{i\omega\rho}\left(\frac{\partial p_A}{\partial z}\right)_{z=0}=0, \quad b<r<a. \quad (67)$$

The boundary conditions at $z=l$ are

$$(p_B)_{z=l}=(p_C)_{z=l}, \quad 0<r<b \quad (68)$$

$$\frac{1}{i\omega\rho}\left(\frac{\partial p_B}{\partial z}\right)_{z=l}=\frac{1}{i\omega\rho}\left(\frac{\partial p_C}{\partial z}\right)_{z=l}, \quad 0<r<b \quad (69)$$

$$\frac{1}{i\omega\rho}\left(\frac{\partial p_C}{\partial z}\right)_{z=l}=0, \quad b<r<c. \quad (70)$$

By using the above boundary conditions and the expressions for the pressure and normal velocity given in Eqs. (53)–(56) and (60)–(63), we obtain the following relations:

$$a^2Y_{A_0}A_0=b^2\frac{B_0^*}{\rho c'}, \quad (71)$$

$$Y_{A_0}A_p=2\beta\frac{J_1(x_p\beta)}{x_pJ_0^2(x_p)}\frac{B_0^*}{\rho c'}+2\beta^2\sum_{n=1}^{\infty}Y_{B_n}B_n^*\frac{T_{pn}(\beta)J_0^2(x_n)}{x_nJ_0^2(x_p)}, \quad (72)$$

$$B_0=A_0+2\sum_{p=1}^{\infty}A_p\frac{J_1(x_p\beta)}{x_p\beta}, \quad (73)$$

$$B_n=2\sum_{p=1}^{\infty}A_p\frac{T_{pn}(\beta)}{x_n}, \quad (74)$$

$$c^2Y_{C_0}\mathfrak{C}_0=b^2\frac{\mathfrak{B}_0^*}{\rho c'}, \quad (75)$$

$$Y_{C_m}\mathcal{C}_m=2\alpha\frac{J_1(x_m\alpha)}{x_mJ_0^2(x_m)}\frac{\mathcal{B}_0^*}{\rho c'}+2\alpha^2\sum_{n=1}^{\infty}Y_{B_n}\mathcal{B}_n^*\frac{T_{mn}(\alpha)J_0^2(x_n)}{x_nJ_0^2(x_m)}, \quad (76)$$

$$\mathcal{B}_0=\mathcal{C}_0+2\sum_{m=1}^{\infty}\mathcal{C}_m\frac{J_1(x_m\alpha)}{x_m\alpha}, \quad (77)$$

$$\mathcal{B}_n=2\sum_{m=1}^{\infty}\mathcal{C}_m\frac{T_{mn}(\alpha)}{x_n}, \quad (78)$$

where

$$\left.\begin{array}{l}\beta=b/a\\ \alpha=b/c\end{array}\right\}. \quad (79)$$

By proceeding in the same way as was done for the discontinuity inductance, it can easily be shown that

$$L(\beta)=\frac{2\rho}{\pi b}\sum_{p=1}^{\infty}\left(\frac{\rho c'}{B_0^*}Y_{A_p}A_p\right)\frac{J_1(x_p\beta)}{x_p^2\beta^2}, \quad (80)$$

$$L(\alpha)=\frac{2\rho}{\pi b}\sum_{m=1}^{\infty}\left(\frac{\rho c'}{\mathcal{B}_0^*}Y_{C_m}\mathcal{C}_m\right)\frac{J_1(x_m\alpha)}{x_m^2\alpha^2}, \quad (81)$$

where $L(\beta)$ is the analogous acoustical inductance at the junction separating regions A and B, and $L(\alpha)$ is the analogous acoustical inductance at the junction separating regions B and C. If Eq. (72) is substituted into (80), and Eq. (76) is substituted into (81), we obtain the following results:

$$L(\beta)=\frac{8\rho}{3\pi^2b}\left\{H(\beta)+\frac{3\pi}{2}\sum_{p=1}^{\infty}\frac{J_1(x_p\beta)}{x_p^2J_0^2(x_p)}\times\sum_{n=1}^{\infty}\left(\frac{\rho c'}{B_0^*}Y_{B_n}B_n^*\right)\frac{T_{np}(\beta)J_0^2(x_n)}{x_n}\right\}, \quad (82)$$

$$H(\beta)=\frac{3\pi}{2}\sum_{p=1}^{\infty}\frac{J_1^2(x_p\beta)}{(x_p\beta)[x_pJ_0(x_p)]^2}, \quad (83)$$

$$L(\alpha)=\frac{8\rho}{3\pi^2b}\left\{H(\alpha)+\frac{3\pi}{2}\sum_{m=1}^{\infty}\frac{J_1(x_m\alpha)}{x_m^2J_0^2(x_m)}\times\sum_{n=1}^{\infty}\left(\frac{\rho c'}{\mathcal{B}_0^*}Y_{B_n}\mathcal{B}_n^*\right)\frac{T_{mn}(\alpha)J_0^2(x_n)}{x_n}\right\}, \quad (84)$$

$$H(\alpha)=\frac{3\pi}{2}\sum_{m=1}^{\infty}\frac{J_1^2(x_m\alpha)}{(x_m\alpha)[x_mJ_0(x_m)]^2}. \quad (85)$$

$H(\alpha)$ and $H(\beta)$ can be found from Fig. 3.

The inductances given by Eqs. (82) and (84) are to be inserted in series with the line at the junction between regions A and B and the junction between regions B and C, respectively. The constriction inductance, which is defined to be the inductance introduced by the change in tube cross section at both ends of the constriction, is given by

$$\mathcal{L}=L(\alpha)+L(\beta). \quad (86)$$

Note that if the double summation is neglected, the constriction inductance can be written as the sum of the discontinuity inductances at each end taken separately. Thus,

$$\mathcal{L}=8\rho/3\pi^2b\{H(\alpha)+H(\beta)\}. \quad (87)$$

It is obvious that the double summation is negligible provided (l/b) is sufficiently large, since all higher order modes excited by one discontinuity will be attenuated before affecting the behavior at the other discontinuity. In the case when (l/b) is small, it is necessary to perform numerical calculations in order to justify Eq. (87). The work required is tedious and lengthy and has not been performed. In similar problems involving parallel plate transmission lines, Whinnery and Jamieson[2] investigated contributions of this kind and found them small.

The analogous inductance of a tube of circular cross section at low frequencies allowing for end effects can now be given. If ρ is the density of the medium, A the cross-sectional area of the tube, and l the actual length of the tube, then

$$L=\frac{\rho l}{A}+(\rho/A)(8b/3\pi)\{H(\alpha)+H(\beta)\} \quad (88)$$

or

$$L=\rho l_e/A, \quad (89)$$

where

$$l_e=l+(8b/3\pi)\{H(\alpha)+H(\beta)\}. \quad (90)$$

The constriction inductance can, therefore, be considered as a correction term to be added to the analogous acoustical inductance of a tube of circular cross section and can be interpreted physically as an increase in the equivalent length of the tube.